Problem with a solution proposed by Arkady Alt , San Jose , California, USA.
Let $D := \{(x,y) \mid x,y \in \mathbb{R}_+, x \neq y$ and $x^y = y^x\}$. (Obvious that $x \neq 1$ and $y \neq 1$).
Find $\sup_{(x,y)\in D}\left(\dfrac{x^{-1}+y^{-1}}{2}\right)^{-1}$

Solution.
Since D is symmetrical then $D = D_< \cup D_>$, where $D_< := \{(x,y) \mid x,y \in D$ and $x < y\}$ and $D_> := \{(x,y) \mid x,y \in D$ and $x > y\}$.
Let $f(x) := x^{\frac{1}{x}}, x > 0$. Since $f'(x) = \dfrac{f(x)}{x^2}(1-\ln x)$ then $f(x)$ strictly increasing on $(0,e]$ and strictly decreasing on $[0,\infty)$ with $\max_{x>0} f(x) = f(e) = e^{\frac{1}{e}}$.
Therefore, noting that $x^y = y^x \Leftrightarrow x^{\frac{1}{x}} = y^{\frac{1}{y}} \Leftrightarrow f(x) = f(y)$, we can conclude that $(x,y) \in D_< \Rightarrow x < e < y$, and $(x,y) \in D_> \Rightarrow y < e < x$. (otherwise if x,y both belong to $(0,e)$ or (e,∞) then, due to monotonicity $f(x)$ equality $f(x) = f(y)$ yields $x = y$, that is the contradiction.
And, also, if $x = e$ then $y > e \Rightarrow f(e) = f(y) < f(e)$).
Also note that if $(x,y) \in D_<$ then $x > 1$. Indeed, since $x^y = y^x \Leftrightarrow$ $y = x\log_x y$, then supposition $x < 1$ implies $0 < \dfrac{y}{x} = \log_x y < 0$, i.e. contradiction.
Hence $(x,y) \in D_< \Rightarrow x \in (1,e),\ y \in (e,\infty)$ and then $(x,y) \in D \Rightarrow x,y \in (1,e) \cup (e,\infty)$.
Let $t := \log_x y - 1$. Then $\log_x y = t+1 \Leftrightarrow y = x^{t+1}$ and $y = x\log_x y \Leftrightarrow$ $y = x(t+1)$.
Hence $y = x^{t+1}$, and, therefore, $x^{t+1} = x(t+1) \Leftrightarrow x^t = t+1 \Leftrightarrow$ $x = (t+1)^{\frac{1}{t}} \Rightarrow y = (t+1)^{1+\frac{1}{t}}$, where $t > 0$ (since $1 < x < y$)
Thus, $D_< = \left\{\left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}}\right) \mid t \in (0,\infty)\right\}$ is set of all non-trivial solution of equation $x^y = y^x$, satisfied $x < y$.
Since $t = \log_x y - 1$ then $(x,y) \in D_> \Rightarrow 1 < y < e < x \Rightarrow$ $1 < y < x \Leftrightarrow \log_x 1 - 1 < \log_x y - 1 < \log_x x - 1 \Leftrightarrow -1 < t < 0$.
Therefore, $D_> = \left\{\left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}}\right) \mid t \in (-1,0)\right\}$ and
$$D = \left\{\left((t+1)^{\frac{1}{t}}, (t+1)^{1+\frac{1}{t}}\right) \mid t \in (-1,0) \cup (0,\infty)\right\}.$$
Note that $\lim_{t\to 0} x = \lim_{t\to 0}(t+1)^{\frac{1}{t}} = e$ and $\lim_{t\to 0} y = \lim_{t\to 0}(t+1)^{1+\frac{1}{t}} = e$.
Since $x < e < y$ then x and y approaches to e from left and right respectively.
We also have $\lim_{t\to\infty} x = \lim_{t\to\infty}(t+1)^{\frac{1}{t}} = 1$ and $\lim_{t\to\infty} y = \lim_{t\to\infty}(t+1)^{1+\frac{1}{t}} = \infty$.

Remark 1.
Correspondence $t \mapsto x(t) = (t+1)^{\frac{1}{t}} : (0,\infty) \to (1,e)$ is one-to-one correspondence, moreover $x(t)$ is strictly decreasing on $(0,\infty)$. Indeed, since
$$(\ln x(t))' = \frac{1}{t(t+1)} - \frac{\ln(1+t)}{t^2} = \frac{1}{t^2}\left(\frac{t}{t+1} - \ln(1+t)\right)$$

suffice to prove that $\frac{t}{t+1} - \ln(1+t)$ is negative for $t > 0$.

But it is so, because $\left(\frac{t}{t+1} - \ln(1+t)\right)' = \frac{1}{(t+1)^2} - \frac{1}{1+t} = \frac{-t}{(t+1)^2}$

and then $\frac{t}{t+1} - \ln(1+t) < \frac{0}{0+1} - \ln(1+0) = 0$.

Thus equation $x = (t+1)^{\frac{1}{t}}$ for any $x \in (1,e)$ always have unique solution $t(x) \in (0,\infty)$ and then $y(x) := (t(x)+1)^{1+\frac{1}{t(x)}}$ is function of x, such that $D_< = \{(x,y(x)) \mid x \in (1,e)\}$.

Let $H(x,y) := \left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1}$.

For $(x,y) \in D_<$ we have $H(x,y) = \frac{2xy}{x+y} = \frac{2xy}{(t+2)x} = \frac{2y}{t+2} =$

$\frac{2(t+1)^{1+\frac{1}{t}}}{t+2} = e^{h(t)}$, where

$h(t) := \ln \frac{2(t+1)^{1+\frac{1}{t}}}{t+2} = \ln \frac{2t+2}{t+2} + \frac{\ln(t+1)}{t}, t \in (0,\infty)$.

Then $\sup\limits_{(x,y)\in D} H(x,y) = \sup\limits_{(x,y)\in D_<} H(x,y) = \sup\limits_{t\in(0,\infty)} e^{h(t)}$.

Note that $h'(t) = -\frac{1}{t^2}\ln(t+1) + \frac{2}{t(t+2)} = \frac{1}{t^2}\left(\frac{2t}{t+2} - \ln(t+1)\right) < 0$.

Indeed, since $\left(\frac{2t}{t+2} - \ln(t+1)\right)' = \frac{4}{(t+2)^2} - \frac{1}{t+1} = -\frac{t^2}{(t+2)^2}$

then $\frac{2t}{t+2} - \ln(t+1)$ is decreasing on $(0,\infty)$ and, therefore,

$\frac{2t}{t+2} - \ln(t+1) < \lim\limits_{t\to 0}\left(\frac{2t}{t+2} - \ln(t+1)\right) = 0$.

Hence, $h(t)$ is decreasing on $(0,\infty)$ and, therefore,

$h(t) < \lim\limits_{t\to 0} h(t) = \lim\limits_{t\to 0}\left(\ln\frac{2t+2}{t+2} + \frac{\ln(t+1)}{t}\right) = \lim\limits_{t\to 0}\ln\frac{2t+2}{t+2} + \lim\limits_{t\to 0}\frac{\ln(t+1)}{t} =$

$0+1 = 1$ yields $\sup\limits_{t>0} h(t) = 1$.

Thus, $\sup\limits_{(x,y)\in D} H(x,y) = e$ or, in form of inequality

$\left(\frac{x^{-1}+y^{-1}}{2}\right)^{-1} < e$, where $x,y > 0, x \neq y$ and $x^y = y^x$.